2020 B

March 9, 2021.

- Parametric curves.

A map from $[a, b] \longrightarrow \mathbb{R}^n$ is called a parametric curve if it is continuous. For $n=3$,

$$\vec{r}(t) = (f(t), g(t), h(t))$$
$$= f(t)\,\hat{i} + g(t)\,\hat{j} + h(t)\,\hat{k} \quad \text{or}$$
$$= (x(t), y(t), z(t))$$
$$= x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}.$$

$\vec{v}(t) \equiv \vec{r}\,'(t)$ is the velocity of $\vec{r}(t)$

$$= x'(t)\,\hat{i} + y'(t)\,\hat{j} + z'(t)\,\hat{k}$$

$$|\vec{v}(t)| = \sqrt{x'^2(t) + y'^2(t) + z'^2(t)}$$

is the speed of $\vec{r}(t)$.

Image a particle moving in space, $\vec{r}(t)$ its position at time t, then $\vec{v}(t)$ is really its velocity.

A parametric curve is C^1 if $x(t)$, $y(t)$, $z(t)$ are C^1-functions. It is regular if $x'(t)\,y'(t)\,z'(t) \neq 0$, ie, $|\vec{v}(t)| > 0$, $\forall t \in (a, b)$. (not nec. at entpoints)

The image of a parametric curve is called a curve.

Some examples.

~ Let $\vec{u}_1$ and $\vec{u}_2$ be two distinct points. The line segment from $\vec{u}_1$ to $\vec{u}_2$ is

$$\vec{r}(t) = \vec{u}_1 + t(\vec{u}_2 - \vec{u}_1), \quad t \in [0, 1].$$

~ The circle $x^2+y^2=r^2$ can be parametrized by

$$\vec{r}(\theta) = r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j}, \quad \theta\in[0,2\pi]$$

$$|\vec{v}(\theta)| = \sqrt{(-r\sin\theta)^2+(r\cos\theta)^2} = r>0 \quad \text{constant speed.}$$

~ The semicircle $x^2+y^2=r^2,\ y\geq 0$, admits 2 parametrizati-

$$\vec{r}(\theta) = r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j}, \quad \theta\in[0,\pi].$$

Next,

$$\vec{r}_1(x) = (x, \sqrt{r^2-x^2}), \quad x\in[-r,r]$$
$$= x\,\hat{i} + \sqrt{r^2-x^2}\,\hat{j}.$$

~ In general, when f is a function over $[a,b]$, its graph is a parametric curve

$$\vec{r}(x) = x\,\hat{i} + f(x)\,\hat{j}$$

$$|\vec{V}(x)| = \sqrt{1+f'^2(x)} > 0 \quad \text{always regular.}$$

- Integration.

  Let f be a continuous function along C. Want to define

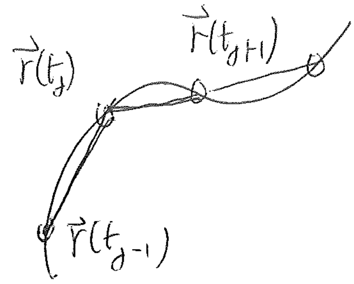
$$\int_C f\,ds.$$

the motivation: Imagine $f\geq 0$ is the density of the thin wire C.

The approximate mass is the sum

$$\sum_{j=1}^{n} f(\vec{r}(t_j^*))\,|\vec{r}(t_j)-\vec{r}(t_{j-1})|$$

$|\vec{r}(t_j) - \vec{r}(t_{j-1})|$ is the length of the line segment connecting $\vec{r}(t_j)$ and $\vec{r}(t_{j-1})$. $t_j^* \in [t_{j-1}, t_j]$



$n=2$

$$|\vec{r}(t_j) - \vec{r}(t_{j-1})| = \sqrt{(x(t_j) - x(t_{j-1}))^2 + (y(t_j) - y(t_{j-1}))^2}$$
$$= \sqrt{x'^2(t_j^{**})(\Delta t_j)^2 + y'^2(t_j^{\star})(\Delta t_j)^2}$$

by the mean-value theorem. $t_j^{**}, t_j^{\star} \in [t_{j-1}, t_j]$.

$$\therefore \text{Sum} = \sum_{j=1}^{n} f(\vec{r}(t_j^*)) \sqrt{x'(t_j^{**})^2 + y'^2(t_j^{\star})}\, \Delta t_j$$
$$\longrightarrow \int_a^b f(\vec{r}(t)) \sqrt{x'(t)^2 + y'^2(t)}\, dt$$

So, we define, for any C in $\mathbb{R}^n$, the <u>line integral</u> of f <u>along</u> C:

$$\int_C f\, ds = \int_a^b f(\vec{r}(t))\, |\vec{v}(t)|\, dt .$$

When $f \geq 0$, this integral give the mass of the wire with density f

when $f = 1$, $\int_C ds$ gives the length of C.

e.g. Evaluate $\int_C f\, ds$, $f = x - 3y^2 + z$,
C: line segment from $(0,0,0)$ to $(1,1,1)$

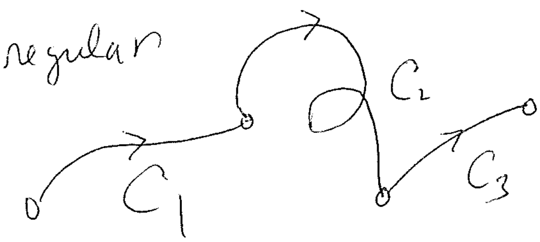
$$\vec{r}(t) = (0,0,0) + t((1,1,1)-(0,0,0))$$
$$= t(\hat{i}+\hat{j}+\hat{k})$$
$$\vec{v}(t) = \hat{i}+\hat{j}+\hat{k}$$
$$|\vec{v}(t)| = \sqrt{3}.$$

$$\therefore \int_C f\,ds = \int_0^1 (t-3t^2+t)\sqrt{3}\,dt = \sqrt{3}\int_0^1 (2t-3t^2)\,dt$$
$$= \sqrt{3}(t^2-t^3)\Big|_0^1$$
$$= 0.$$

A set of curves $C_1, C_2, .. C_n$ satisfy: the endpoint of C_{j-1} is the starting point of C_j, $\forall j$. Then we can put them together to form a piecewise curve:

$$C = C_1 + \cdots + C_n.$$

Call it piecewise regular if each C_j is regular



Define

$$\int_C f\,ds = \sum_{j=1}^{n} \int_{C_j} f\,ds$$

We allow in C some non-smooth points.

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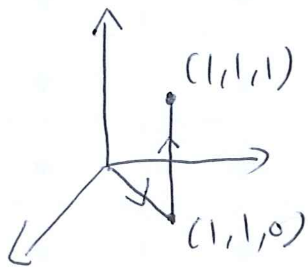
e.g. Find $\int_C f\,ds$ when $C = C_1 + C_2$, $f = x - 3y^2 + z$

C_1 line segment $(0,0,0)$ to $(1,1,0)$

C_2 line segment $(1,1,0)$ to $(1,1,1)$



C_1 : $\vec{r}_1(t) = (0,0,0) + t((1,1,0)-(0,0,0))$
$= (t,t,0)$, $t \in [0,1]$

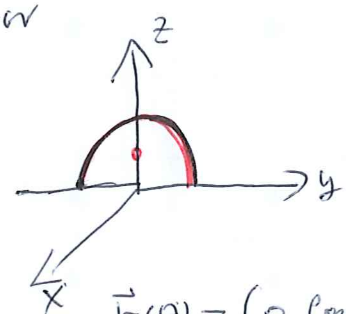
C_2 $\vec{r}_2(t) = (1,1,0) + t((1,1,1)-(1,1,0))$
$= (1,1,t)$, $t \in [0,1]$

$$\int_{C_1} f\,ds = \int_0^1 (t - 3t^2 - 0)\sqrt{2}\,dt = -\frac{\sqrt{2}}{2}$$

$$\int_{C_2} f\,ds = \int_0^1 (1 - 3 + t)\sqrt{1}\,dt = -\frac{3}{2}$$

$$\therefore \int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds = -\frac{\sqrt{2}}{2} - \frac{3}{2}$$

e.g. Find the mass and center of mass for the arc $y^2 + z^2 = 1$, $z \geq 0$, $\delta = 2 - z$.



$\vec{r}(\theta) = (0, \cos\theta, \sin\theta)$
$\theta \in [0, \pi]$
$|\vec{r}'(\theta)| = 1$

$$M = \int_C \delta\,ds$$
$$= \int_0^\pi (2 - \sin\theta)\cdot 1 \cdot d\theta = 2\pi - 2.$$

$$M_{xy} = \int_C z\delta\,ds$$
$$= \int_0^\pi (\sin\theta)(2 - \sin\theta)\,1\,d\theta = \frac{8-\pi}{2}$$

$$\vec{c} = \left(0, 0, \frac{8-\pi}{2} \times \frac{1}{2\pi - 2}\right) \approx (0, 0, 0.57).$$

Now, we show that $\int_C f\,ds$ is indept of the parametrization.

<u>Theorem</u> Let $\vec{r}:[a,b]\to C$, $\vec{c}:[\alpha,\beta]\to C$ are 2 parametrization of C (ie, 1-1 onto, regular). Then

$$\int_a^b f(\vec{r}(t))|\vec{r}\,'(t)|\,dt = \int_\alpha^\beta f(\vec{c}(\tau))|\vec{c}\,'(\tau)|\,d\tau.$$

Pf. $\forall t\in[a,b]$, $\vec{r}(t)$ is a pt on C, so there is a unique τ s.t. $\vec{r}(t)=\vec{c}(\tau)$. The correspondence $t\mapsto\tau$ set up a map $\tau=\varphi(t)$ from $[a,b]$ 1-1 onto $[\alpha,\beta]$.
there are two cases
ⓐ $\varphi(a)=\alpha$, $\varphi(b)=\beta$, ⓑ $\varphi(a)=\beta$, $\varphi(b)=\alpha$.

Diff. the relation, $\vec{r}(t)=\vec{c}(\varphi(t))$,

$$\vec{r}\,'(t)=\vec{c}\,'(\tau)\,\varphi'(t)$$

$$|\vec{r}\,'(t)|=|\vec{c}\,'(\tau)|\,|\varphi'(t)|.$$

Since $\vec{r}$ and $\vec{c}$ are regular, $|\vec{r}\,'(t)|, |\vec{c}\,'(\tau)|>0$, so

$$|\varphi'(t)|>0,\ \forall t.$$

In ⓐ $\varphi'(t)>0$, in ⓑ $\varphi'(t)<0$.

Using the change of variables $\tau=\varphi(t)$, $d\tau=\varphi'(t)\,dt$

In ⓐ
$$\int_a^b f(\vec{r}(t))|\vec{r}\,'(t)|\,dt=\int_\alpha^\beta f(\vec{c}(\tau))|\vec{c}\,'(\tau)||\varphi'(t)|\frac{1}{\varphi'(t)}\,dt$$
$$=\int_\alpha^\beta f(\vec{c}(\tau))|\vec{c}\,'(\tau)|\,dt \quad (\because \varphi'(t)>0)$$

In ⓑ
$$\int_a^b f(\vec{r}(t))|\vec{r}\,'(t)|\,dt=\int_\beta^\alpha f(\vec{c}(\tau))|\vec{c}\,'(\tau)||\varphi'(t)|\frac{1}{\varphi'(t)}\,dt$$
$$=-\int_\beta^\alpha f(\vec{c}(\tau))|\vec{c}\,'(\tau)|\,d\tau \quad (\varphi'(t)<0)$$

$$= \int_{\alpha}^{\beta} f(\vec{c}(\tau))\,|\vec{c}\,'(\tau)|\,d\tau . \quad \text{done } \#$$

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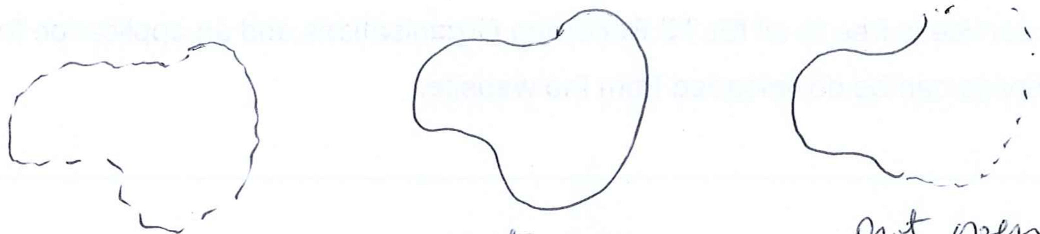
Now, consider the integration of a v.f. along a curve.

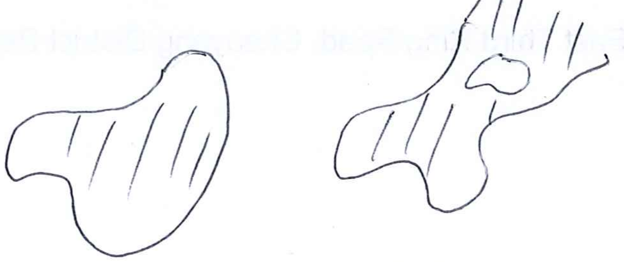
Usually a vector field is defined in an open region.

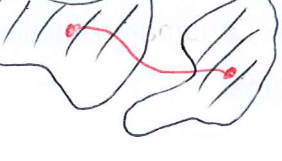
A set G is called an open set if $\forall \vec{u} \in G$, there is a ball centered at $\vec{u}$, $B(\vec{u})$, such that $B(\vec{u}) \subset G$. Roughly speaking, a set is open if it contains no boundary points.



open   not open.   Not open

A set G is a region if any 2 pts in G can be connected by a curve inside G.





not a region

region   region (unbounded)

A vector field in an open region G is

$$\vec{F}(\vec{u}) = (M(\vec{u}), N(\vec{u}), P(\vec{u}))$$
$$= M(\vec{u})\,\hat{i} + N(\vec{u})\,\hat{j} + P(\vec{u})\,\hat{k},$$

or

$$\vec{F}(x,y,z) = M(x,y,z)\,\hat{i} + N(x,y,z)\,\hat{j} + P(x,y,z)\,\hat{k}$$

where M, N, P are functions in G. It is C^1-v.f. if M, N, P are C^1-functions.

Some examples

- velocity of a fluid in river
- gravitational force field

magnitude $g\dfrac{Mm}{r^2}$,

direction between M, m

Taking the position of M as the origin,

$$\vec{F}(x,y,z) = \frac{gMm}{(x^2+y^2+z^2)} \times \frac{-(x,y,z)}{(x^2+y^2+z^2)^{1/2}}$$

$$= -gMm\,\frac{(x\hat{i} + y\hat{j} + z\hat{k})}{(x^2+y^2+z^2)^{3/2}}.$$

$\vec{F}$ is defined in the open region

$$\{(x,y,z) : (x,y,z) \neq (0,0,0)\}$$

- gradient vector fields.

Let f be a function in G. Its gradient

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

$$= \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

is a vector field in G. It is C^1 if f is C^2.

The gravitational v.f is a gradient one.

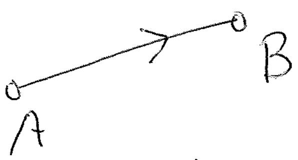
$$\varphi = \frac{gMm}{(x^2+y^2+z^2)^{1/2}}.$$

You can check

$$\nabla\varphi = -gMm\,\frac{x\hat{i}+y\hat{j}+z\hat{k}}{(x^2+y^2+z^2)^{3/2}}.$$

Integration of a vector field.

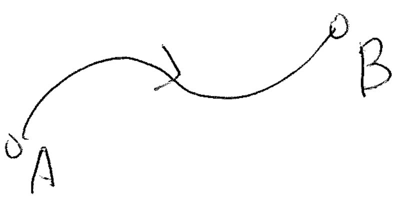
Recall in physics. An object moves from A to B



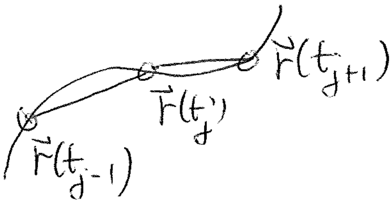
under a constant v.f $\vec{F}$. the work done is

$$\vec{F}\cdot(\vec{B}-\vec{A}).$$

Now, let $\vec{F}$ be non-constant and A goes to B along a curve C. How to define its work done.



Let $\vec{r}(t): t\in[a,b] \to C$ be a parametrization of C, $\vec{r}(a)=A$, $\vec{r}(b)=B$. Let $a=t_0<t_1<\dots<t_n=b$.



Moving from $\vec{r}(t_{j-1})$ to $\vec{r}(t_j)$ along C is approx. along the line segment, and the force is approx. $\vec{F}(\vec{r}(t_{j-1}))$.

Hence, the work done is approx.

$$\sum_j \vec{F}(\vec{r}(t_{j-1})) \cdot (\vec{r}(t_j) - \vec{r}(t_{j-1}))$$

$$= \sum_j \vec{F}(\vec{r}(t_{j-1})) \cdot \vec{r}\,'(t_j^*) \Delta t_j , \qquad t_j^* \text{ some mean value}$$

$$\longrightarrow \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt .$$

We define the integral of $\vec{F}$ along C to be

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt .$$

We also use the notation

$$\int_C \vec{F} \cdot d\vec{r} = \int_C M\,dx + N\,dy + P\,dz .$$

e.g. Find the work done $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = z\hat{i} + xy\hat{j} - y^2\hat{k}$

$C : \vec{r}(t) = t^2\hat{i} + t\hat{j} + \sqrt{t}\hat{k}$, $t \in [0,1]$.

$$\vec{F}(\vec{r}(t)) = \sqrt{t}\,\hat{i} + t^3\hat{j} - t^2\hat{k}$$

$$\vec{r}\,'(t) = 2t\,\hat{i} + \hat{j} + \tfrac{1}{2} t^{-1/2}\hat{k}$$

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = \sqrt{t} \times 2t + t^3 \times 1 + (-t^2)\tfrac{1}{2} t^{-1/2}$$

$$= \frac{3}{2} t^{3/2} + t^3 .$$

$\therefore$ work done $= \int_0^1 \left(\frac{3}{2} t^{3/2} + t^3\right) dt$

$= 17/20$ #